Journal of Approximation Theory **103**, 183–205 (2000)
Article ID jath.1999.3425, available online at http://www.idealibrary.com on IDEAL®

On a Polynomial Inequality of Paul Erdős[1]

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Communicated by Peter B. Borwein

Received December 4, 1998; accepted September 16, 1999

Let f be a real polynomial having no zeros in the open unit disk. We prove a sharp evaluation from above for the quantity $\|f'\|_\infty/\|f\|_p$, $0 \leqslant p < \infty$. The extremal polynomials and the exact constants are given. This extends an inequality of Paul Erdős [7].

Key Words: polynomial inequalities of Erdős type, exact constants, exact asymptotic.

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

For any continuous function f: $[-1, 1] \to \mathbf{C}$ and $p \in (0, \infty)$ let

$$\|f\|_p := \left(\frac{1}{2}\int_{-1}^{1} |f(x)|^p \, dx\right)^{1/p};$$

besides, let

$$\|f\|_\infty := \max_{-1 \leqslant x \leqslant 1} |f(x)|.$$

It is known that $\|f\|_p$ tends to the limit

$$\exp\left(\frac{1}{2}\int_{-1}^{1} \log |f(x)| \, dx\right)$$

when $p \to 0$. This is exactly the value given to the functional $\|f\|_p$ when $p = 0$.

Let f be a real polynomial. If f has no zeros in the open unit disk then $\|f'\|_\infty$ can be best possible estimated from above by $\|f\|_p$, $0 \leqslant p \leqslant \infty$. To the best of our knowledge the first result of this type for polynomials with restricted zeros was obtained by Erdős [7] in the case $p = \infty$.

[1] Research partially supported by Sofia University grant No. 314/99.

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0021-9045/00 $35.00

THEOREM A. *Let f be a real polynomial of degree at most n such that $\|f\|_\infty = 1$. If the zeros of f are all real and lie on $\mathbf{R}\backslash(-1, 1)$ then*

$$\|f'\|_\infty \leqslant \left[\frac{1}{2}\left(1-\frac{1}{n}\right)^{-n+1}\right] n.$$

Equality is possible only at -1, $+1$ and it holds only for

$$f_{ex.,\,1}(x) := \frac{n^n}{2^n(n-1)^{n-1}}(1+x)(1-x)^{n-1}$$

and

$$f_{ex.,\,2}(x) := \frac{n^n}{2^n(n-1)^{n-1}}(1+x)^{n-1}(1-x).$$

It can be seen that the exact asymptotic of $\|f'\|_\infty/\|f\|_\infty$ with respect to the polynomial degree is n. This is smaller than n^2, which is the exact asymptotic in the corresponding inequality for polynomials without restrictions on the zeros, see [8].

Without any restriction we can assume that each polynomial, belonging to the class introduced in Theorem A, is positive on $(-1, 1)$. By using a wider class of polynomials (see Remark 8)

$$\pi_n := \left\{f : f(x) = \sum_{k=0}^{n} A_k(1+x)^k(1-x)^{n-k},\ A_k \geqslant 0,\ k=0, 1, ..., n\right\}$$

introduced by Bernstein (see [3, 4]) an extension of Theorem A is due to Sheick [12].

THEOREM B. *If $f \in \pi_n$, then*

$$\|f'\|_\infty \leqslant \frac{e}{2} n \|f\|_\infty.$$

On the basis of a sharp point-wise bound for $|f'(x)|/\|f\|_\infty$ at an arbitrarily prescribed point x on the unit interval the next theorem is given in [1].

THEOREM C. *If f is a real polynomial of degree at most n such that $\|f\|_\infty = 1$ and $f(z) \neq 0$, $|z| < 1$, then*

$$\|f'\|_\infty \leqslant \left[\frac{1}{2}\left(1-\frac{1}{n}\right)^{-n+1}\right] n$$

with a case of equality only for $f_{ex.,\,1}$ and $f_{ex.,\,2}$.

Another contribution to this subject can be seen in [2], where the following theorem has been proved.

THEOREM D. *If f is a real polynomial of degree at most n having at most k zeros in the open unit disk, then there exists an absolute constant c, such that*

$$\|f'\|_\infty \leqslant cn(k+1)\, \|f\|_\infty .$$

Note, that the estimate is sharp up to the best possible constant. A short proof of the above theorem based on D. Newman inequality [10] for Münz polynomials and Meissner's representation (see Remark 8 of this paper) can be seen in [6]. The best constants case is known only in the cases $k=0$, [1, 7, 12] and $k=n$, [8].

Let $q_{n,k}(x)=(1+x)^k\,(1-x)^{n-k}$ and $q_{n,k,*}(x)=(n^n q_{n,k}(x))/(2^n k^k (n-k)^{n-k})$. Note that $\|q_{n,k,*}\|_\infty=1$. The next Erdős–Nikolskii type inequality between different metrics is proved in [5].

THEOREM E. *Let f be a polynomial of degree at most n with real coefficients and having no zeros in the open unit disk. Suppose, in addition, that f has zeros of multiplicity at least μ at -1 and 1, where $0\leqslant\mu\leqslant[n/2]$. If f is not a constant multiple by $q_{n,\mu}$ or $q_{n,n-\mu}$, then*

$$\|f\|_p > \|q_{n,\mu,*}\|_p\, \|f\|_\infty, \qquad 0\leqslant p<\infty.$$

As a corollary of Theorem C and Theorem E the best possible estimate of $\|f'\|_\infty/\|f\|_p$ from above is given in [5] but under the additional boundary assumption

$$f(-1)=f(1)=0\,.$$

COROLLARY A. *Let f be a real polynomial of degree at most n, such that $f(-1)=f(1)=0$ and $f(z)\neq 0$ for $|z|<1$. If f is not a constant multiple by $q_{n,1}$ or $q_{n,n-1}$ then*

$$\|f'\|_\infty < \frac{\|q'_{n,1}\|_\infty}{\|q_{n,1}\|_p}\, \|f\|_p, \qquad 0\leqslant p<\infty.$$

Now a natural question arises, stated by Professor Q. I. Rahman in a joint discussion with the first named author: If it is possible that the boundary conditions $f(-1)=f(1)=0$ can be removed from the statement of Corollary A. It is logical to believe, following [7], that these boundary conditions are superfluous in the statement of Corollary A at least for $p>1$.

We will answer the above mentioned question by giving the exact solution of the extremal problem, given in Corollary A, without the boundary condition $f(-1)=f(1)=0$. Our method is based on a technical refinement of a variational approach, given in [5].

Let $\mathscr{P}_n$ denote the class of all real polynomials of degree at most n having no zeros in the open unit disk, i.e. $f(z)\neq 0$, $|z|<1$. Then our extremal problem reads as follows. Find

$$\sup_{f\in\mathscr{P}_n} \frac{\|f'\|_\infty}{\|f\|_p}, \qquad 0\leqslant p<\infty. \tag{1}$$

Remark 1. Without any restriction in the process of our considerations we can assume that $\mathscr{P}_n$ consists of polynomials which are positive on $(-1, 1)$.

Remark 2. We will show that the boundary conditions $f(-1)=f(1)=0$ in Corollary A are superfluous when $p>1$ (see Theorem 1). This means that the extremal polynomials of (1) has to satisfy $f(-1)=f(1)=0$ in the case $p>1$. If $0\leqslant p<1$ and we do not put the boundary condition $f(-1)=f(1)=0$, (see Theorem 1) then the extremal polynomials of (1) will differ from the extremal polynomials of the corresponding to (1) problem with the boundary conditions $f(-1)=f(1)=0$ (this case is settled in Corollary A).

Remark 3. In the process of solving the problem (1) we will see that in the case $0\leqslant p\leqslant 1$ new effects appears. Namely, the extremal polynomials of problem (1) do not satisfy the boundary condition $f(-1)=f(1)=0$, when $0\leqslant p<1$. In the case $p=1$ we will see that the problem (1) has two classes of extremal polynomials. One of them satisfies $f(-1)=f(1)=0$ whereas the other does not. In the case $p>1$ the extremal polynomials of (1) must satisfy the condition $f(-1)=f(1)=0$ and this means that the condition $f(1)=f(-1)=0$ is superfluous in the statement of Corollary A for $p>1$.

Problem. For a fixed p, $0\leqslant p<\infty$, find the exact value and the extremal polynomials of (1).

2. AUXILIARY RESULTS

The solution of the problem (1) is based on a technical refinement of a variational approach that is described in [5] and will be presented as a sequence of lemmas.

LEMMA 1. *The problem* (1) *possesses an extremal polynomial. In other words*

$$\sup_{f \in \mathscr{P}_n} \frac{\|f'\|_\infty}{\|f\|_p} = \max_{f \in \mathscr{P}_n} \frac{\|f'\|_\infty}{\|f\|_p}$$

Proof. From Theorem A and Theorem C we have

$$\|f'\|_\infty \leqslant \frac{1}{2}\left(1-\frac{1}{n}\right)^{-n+1} n\,\|f\|_\infty \leqslant C(n, p)\,\|f\|_p,$$

so

$$\sup_{f \in \mathscr{P}_n} \frac{\|f'\|_\infty}{\|f\|_p} \leqslant C(n, p) < +\infty.$$

Let f_k be a sequence of polynomials from $\mathscr{P}_n$ such that

$$\frac{\|f'_k\|_\infty}{\|f_k\|_p} \geqslant \sup_{f \in \mathscr{P}_n} \frac{\|f'\|_\infty}{\|f\|_p} - \frac{1}{k}.$$

Consider polynomials $g_k := f_k/\|f_k\|_p$. It is evident that $\|g_k\|_p = 1$ for $k = 1, 2, \ldots$.

From the above-mentioned papers it follows that $\|g_k\|_\infty \leqslant C(n, p)$ for $k = 1, 2, \ldots$. We choose from the sequence $\{g_k\}_{k=1}^{\infty}$ a locally uniformly convergent subsequence and denote it by $\{g_k\}_{k=1}^{\infty}$.

The locally uniform limit g of $\{g_k\}_{k=1}^{\infty}$ must be a polynomial of degree at most n. By Lebesgue's dominated convergence theorem it follows that $\|g\|_p = 1$ and from here the limit function g is not identically zero. Hurwitz's theorem shows that $g \in \mathscr{P}_n$. In particular $g(x) \neq 0$ on $-1 < x < 1$.

The degree of our polynomials $g_k(x)$ for $k = 1, 2, \ldots$ is fixed and the locally uniform convergence is an invariant property with respect to differentiation. This means that $g'_k \to g'$ locally uniformly, when $k \to \infty$.

From the above considerations it is easily seen that

$$\lim_{k \to \infty} \|g_k\|_p = \|g\|_p, \qquad \lim_{k \to \infty} \|g'_k\|_\infty = \|g'\|_\infty.$$

Hence we have

$$\sup_{f \in \mathscr{P}_n} \frac{\|f'\|_\infty}{\|f\|_p} \geqslant \frac{\|g'\|_\infty}{\|g\|_p} = \|g'\|_\infty = \lim_{k \to \infty} \frac{\|g'_k\|_\infty}{\|g_k\|_p} = \lim_{k \to \infty} \frac{\|f'_k\|_\infty}{\|f_k\|_p} \geqslant \sup_{f \in \mathscr{P}_n} \frac{\|f'\|_\infty}{\|f\|_p}$$

which gives

$$\frac{\|g'\|_\infty}{\|g\|_p} = \sup_{f \in \mathcal{P}_n} \frac{\|f'\|_\infty}{\|f\|_p}.$$

The proof is completed. The conclusion is that an extremal polynomial exists.

Let us denote by ρ_n the subclass of $\mathcal{P}_n$ consisting of polynomials having only real zeros, i.e.

$$\rho_n = \{f\colon f \in \mathcal{P}_n, f \text{ has only real zeros}\}$$

The next lemma indicates that while looking for extremal polynomials of (1) we only need to examine those polynomials of $\mathcal{P}_n$ whose all zeros are real.

LEMMA 2. *We have*

$$\sup_{f \in \mathcal{P}_n} \frac{\|f'\|_\infty}{\|f\|_p} = \sup_{f \in \rho_n} \frac{\|f'\|_\infty}{\|f\|_p}.$$

Proof. If $deg(f) \leqslant 1$ the statement of the lemma is trivial. Let $deg(f) \geqslant 2$ and let f have at least one non-real zero. Without any restriction we may assume that f is positive on $(-1, 1)$. Since f is real, f has a pair of conjugate zeros, so $f(z) = g(z)(z-a-ib)(z-a+ib)$, $b \in R$, $b \neq 0$.

Let $\xi \in [-1, 1]$ be such that

$$\|f'\|_\infty = |f'(\xi)|.$$

Consider an auxiliary polynomial

$$f_\varepsilon(z) = f(z) - \varepsilon g(z)(z-\xi)^2, \qquad \varepsilon \text{ sufficiently small and positive.}$$

Note that $f'_\varepsilon(\xi) = f'(\xi)$ and from here

$$\|f'_\varepsilon\|_\infty \geqslant |f'_\varepsilon(\xi)| = |f'(\xi)| = \|f'\|_\infty.$$

Now $f_\varepsilon(z)$ can be represented as follows

$$f_\varepsilon(z) = g(z)((1-\varepsilon)\, z^2 + 2(\xi\varepsilon - a)\, z + a^2 + b^2 - \varepsilon\xi^2)$$

and the quadratic

$$(1-\varepsilon)\, z^2 + 2(\xi\varepsilon - a)\, z + a^2 + b^2 - \varepsilon\xi^2$$

must have a pair of conjugate zeros for ε sufficiently small ($\varepsilon > 0$) because $b \neq 0$. So

$$f_\varepsilon(z) = (1-\varepsilon)\, g(z)(z - z_{1,\varepsilon})(z - \bar{z}_{1,\varepsilon}).$$

We have

$$|z_{1,\varepsilon}|^2 = z_{1,\varepsilon}\bar{z}_{1,\varepsilon} = \frac{a^2 + b^2 - \varepsilon\xi^2}{1-\varepsilon} \geqslant 1.$$

and $z_{1,\varepsilon}$, $\bar{z}_{1,\varepsilon}$ do not belong to the open unit disk. On the other hand $\max_{x \in [-1,1]} |f(x)| = f(x^*) > 0$ and for $\varepsilon > 0$ and sufficiently small $f_\varepsilon(x^*) > 0$. Hence $f_\varepsilon(x) > 0$ for $-1 < x < 1$. The conclusion is that $f_\varepsilon(z) \in \mathscr{P}_n$.

By assumption $f > 0$ on $(-1, 1)$ and from here $g > 0$ on $(-1, 1)$. It follows that

$$0 < f_\varepsilon(x) < f(x), \qquad x \in (-1, 1) \backslash \{\xi\}$$

and

$$\|f_\varepsilon\|_p < \|f\|_p.$$

Thus

$$\frac{\|f'\|_\infty}{\|f\|_p} < \frac{\|f'_\varepsilon\|_\infty}{\|f_\varepsilon\|_p}$$

and this ends the proof of the lemma.

Remark 4. Note that if $b = 0$ the above considerations do not work and we may have a pair of real zeros and at least one of them can be in the open unit disk.

If $deg(f) = 0$ then $\sup_{f \in \mathscr{P}_0} \|f'\|_\infty / \|f\|_p = 0$ so the problem is trivial and each non-zero constant polynomial is a solution of our problem. If $deg(f) = 1$ then it is evident that the only extremal polynomials are $c(x+1)$ and $c(1-x)$, $c \in \mathbf{R}$, $c \neq 0$.

In what follows we assume $n \geqslant 2$.

LEMMA 3. *If* $f \in \rho_n$ *and* $f(-1)\, f(1) \neq 0$ *then* f *cannot be extremal.*

Proof. Without any restriction let $f > 0$ on $(-1, 1)$, $f \in \rho_n$ and let

$$\min_{x \in [-1,1]} f(x) = \min[f(-1), f(1)] > 0.$$

Consider the polynomial $f_\varepsilon(z) := f(z) - \varepsilon$, for ε sufficiently small ($\varepsilon > 0$).

Let $\mathbf{C_1} = \{z : z \in \mathbf{C}, |z| \leqslant 1\}$ be the unit disk in the complex plane and $x_1, x_2, ..., x_n$ be the zeros of f.

Since $\min_{1 \leqslant i \leqslant n} |x_i| > 1$, then $\min_{\{i=1, 2, ..., \text{ and } z \in C_1\}} |x_i - z| > 0$ and from this and from Hurwitz's theorem we have $f_\varepsilon(z) \in \mathscr{P}_n$ for $\varepsilon > 0$ and sufficiently small. Note that $f_\varepsilon(z)$ may have complex zeros. If $\varepsilon \leqslant \min[f(-1), f(1)]$ then

$$0 < f_\varepsilon(x) < f(x), \qquad -1 < x < 1$$

and

$$f'_\varepsilon(x) = f'(x).$$

It is easily checked that

$$\frac{\|f'\|_\infty}{\|f\|_p} < \frac{\|f'_\varepsilon\|_\infty}{\|f_\varepsilon\|_p}, \qquad f_\varepsilon \in \mathscr{P}_n$$

and this completes the proof.

LEMMA 4. *If $f(x) \in \rho_n$ and f possesses at least two zeros in $R \backslash [-1, 1]$ counting their multiplicities then $f(x)$ cannot be an extremal polynomial of* (1).

Proof. Let ξ be a point of $[-1, 1]$ where $|f'(x)|$ attains the maximum value($\|f'\|_\infty = |f'(\xi)|$). First we observe that f cannot have zeros in $(-\infty, -1)$ and $(1, \infty)$ at the same time. Suppose it does. Let λ_- be the smallest zero of f and λ_+ be the largest one. It is easily seen that for all small $\varepsilon > 0$ the polynomial

$$f_\varepsilon(z) := f(z) + \frac{\varepsilon f(z)}{(z - \lambda_-)(z - \lambda_+)} (z - \xi)^2$$

belongs to ρ_n and $0 < f_\varepsilon(x) < f(x)$ for all $x \in (-1, 1) \backslash \{\xi\}$. On the other hand

$$\|f'_\varepsilon\|_\infty \geqslant |f'_\varepsilon(\xi)| = |f'(\xi)| = \|f'\|_\infty$$

and we see that

$$\frac{\|f'\|_\infty}{\|f\|_p} < \frac{\|f'_\varepsilon\|_\infty}{\|f_\varepsilon\|_p}.$$

Assume that f has no zeros in $(-\infty, -1)$. We claim that f cannot have two or more zeros in $(1, \infty)$ counting their multiplicities. Suppose it does.

Let λ_1 be the largest zero of f and λ_2 the largest but one. If λ_1 is double zero then $\lambda_1 = \lambda_2$.

It is geometrically evident that for all small $\varepsilon > 0$, the polynomial

$$f_\varepsilon(z) := f(z) - \frac{\varepsilon f(z)}{(z-\lambda_1)(z-\lambda_2)} (z-\xi)^2$$

belongs to ρ_n and $0 < f_\varepsilon(x) < f(x)$ for all $x \in (-1, 1) \backslash \{\xi\}$.

From here we clearly have

$$\frac{\|f'\|_\infty}{\|f\|_p} < \frac{\|f'_\varepsilon\|_\infty}{\|f_\varepsilon\|_p}.$$

The proof is completed.

We have proved that if f is extremal for the problem (1) it must be of the following form

$$f(x) = c(1+tx)(1-x)^j (1+x)^k,$$
$$\text{where} \quad |t| \leqslant 1, \quad k+j \leqslant n-1, \quad c \in \mathbf{R}, \quad c \neq 0.$$

By using an analogous variational construction one may show that $j+k = n-1$ if f is extremal. The point ξ is chosen by analogy such that

$$\|f'\|_\infty = |f'(\xi)|.$$

The above considerations can be summarized in the following lemma.

LEMMA 5. *If f is an extremal polynomial of the problem* (1) *then f must be of the form*

$$f(x) = c(1+tx)(1-x)^{n-k-1} (1+x)^k,$$

where $-1 \leqslant t \leqslant 1$, $0 \leqslant k \leqslant n-1$, $c \in \mathbf{R}$, $c \neq 0$.

Remark 5. Lemma 3 is a corollary of Lemma 5, note that $n \geqslant 2$. Lemma 5 shows that while searching for an extremal polynomials of the problem (1) we need only examine those from the class

$$e_n := \{f \colon f(x) = c(1+x)^k (1-x)^{n-k-1} (1+tx);$$
$$0 \leqslant k \leqslant n-1, \ -1 \leqslant t \leqslant 1, \ c > 0\}.$$

Note that without any restriction we can suppose that $f(x) > 0$ on $(-1, 1)$.

Let

$$e_{n,1} := \{f : f \in e_n;\ f(1) = 0,\ f(-1) > 0\}$$
$$e_{n,-1} := \{f : f \in e_n;\ f(-1) = 0,\ f(1) > 0\}$$
$$e_{n,-1,1} := \{f : f \in e_n;\ f(1) = f(-1) = 0\}.$$

We have

$$\max_{f \in \mathscr{P}_n} \frac{\|f'\|_\infty}{\|f\|_p} = \max\left(\max_{f \in e_{n,-1}} \frac{\|f'\|_\infty}{\|f\|_p},\ \max_{f \in e_{n,1}} \frac{\|f'\|_\infty}{\|f\|_p},\ \max_{f \in e_{n,-1,1}} \frac{\|f'\|_\infty}{\|f\|_p}\right). \tag{2}$$

The above formula shows that we can divide our problem into 3 problems.
The first one,

$$\max_{f \in e_{n,-1,1}} \frac{\|f'\|_\infty}{\|f\|_p}, \tag{3}$$

is the problem which is given by Corollary A, (see [5, Corollary 2]). The extremal polynomials in this case have the form

$$c(1-x)(1+x)^{n-1}, \qquad \text{and} \qquad c(1+x)(1-x)^{n-1}, \quad c > 0.$$

It is obvious that if $f(x)$ is extremal for the problem

$$\max_{f \in e_{n,-1}} \frac{\|f'\|_\infty}{\|f\|_p}$$

then $f(-x)$ is extremal for

$$\max_{f \in e_{n,1}} \frac{\|f'\|_\infty}{\|f\|_p}.$$

and vice versa.

We make the conclusion that to find the extremals of problem (1) we need only study the problem

$$\max_{f \in e_{n,1}} \frac{\|f'\|_\infty}{\|f\|_p}. \tag{4}$$

The class $e_{n,1}$ consists of all polynomials of the form

$$e_{n,1} := \{f : f(x) = c(1+tx)(1-x)^{n-1},\ c > 0,\ -1 \leqslant t < 1\}.$$

LEMMA 6. *If* $t \in (-1, (n-2)/n]$ *then the polynomial* $f(x) = c(1+tx)(1-x)^{n-1}$, $c > 0$ *cannot be extremal for the problem* (1).

Proof. For clarity we divide the proof into four cases with respect to the parameter t.

(a) Let first $t \in (-1, 0)$. By using the polynomial

$$p_{-1}(x) := \frac{(1-x)^{n-1}}{2^n} [(n-1)x + n + 1]$$

$p_{-1}(-1) = 1$, $p'_{-1}(-1) = 0$ and $p_{-1}(x) \geqslant 0$ for $-1 \leqslant x \leqslant 1$ we construct a variational polynomial

$$f_\varepsilon(x) := f(x) - \varepsilon p_{-1}(x).$$

It is geometrically evident that $f_\varepsilon(x)$ must have one real zero greater than 1 and smaller than $-1/t$ for a sufficiently small $\varepsilon > 0$.

So $f_\varepsilon(x) \in e_{n,1}$ for a sufficiently small $\varepsilon > 0$ and $0 < f_\varepsilon(x) < f(x)$ for $-1 \leqslant x < 1$.

On the other hand $\|f'\|_\infty = |f'(-1)|$ and $\|f'_\varepsilon\|_\infty \geqslant |f'_\varepsilon(-1)| = |f'(-1)| = \|f'\|_\infty$.

It follows that

$$\frac{\|f'_\varepsilon\|_\infty}{\|f_\varepsilon\|_p} > \frac{\|f'\|_\infty}{\|f\|_p}$$

and from here the polynomial $f(x) = c(1+tx)(1-x)^{n-1}$ cannot be extremal for $t \in (-1, 0)$.

(b) Let now $t = 0$, so $f(x) = c(1-x)^{n-1}$. Consider again

$$f_\varepsilon(x) := c(1-x)^{n-1} - \varepsilon p_{-1}(x) = (-1)^n \frac{\varepsilon(n-1)}{2^n} x^n + \sum_{k=0}^{n-1} c_k x^k. \tag{5}$$

Without any restriction let us assume n odd. The case n even can be treated by analogy. For a fixed $x_0 > 1$ we can choose ε sufficiently small ($\varepsilon > 0$) such that $f_\varepsilon(x_0) > 0$. From here and the representation (5) it follows $f_\varepsilon(x)$ has a real zero greater than 1 because $\lim_{x \mapsto +\infty} f_\varepsilon(x) = -\infty$. So $f_\varepsilon \in e_{n,1}$ and $0 < f_\varepsilon(x) < f(x)$, $-1 \leqslant x < 1$. We end this case by the same arguments as in the case (a).

The cases (a) and (b) complete the proof of Lemma 6 when $n = 2$. Let $n \geqslant 3$.

(c) Let $n \geqslant 3$ and let $t \in (0, (n-2)/(n+2)]$. Consider $f(x) = c(1+tx)(1-x)^{n-1}$, $c > 0$.

The polynomial $f'(x)$ has one local extremum at $x^*(t) = (2t - n + 2)/nt$. For $t \in (0, (n-2)/(n+2)]$, $x^*(t) \leqslant -1$. From here

$$\max_{x \in [-1, 1]} |f'(x)| = |f'(-1)|$$

and we can proceed as in the case (a).

It is geometrically evident that

$$f_\varepsilon(x) := f(x) - \frac{\varepsilon}{2^n} (1-x)^{n-1} [(n-1)\, x + n + 1]$$

$$= \left[ct - \frac{(n-1)}{2^n} \varepsilon \right] (-1)^{n-1} x^n + \sum_{k=0}^{n-1} c_k x^k$$

will have one real zero less than -1 if ε is sufficiently small ($\varepsilon > 0$) such that $f_\varepsilon(-1) > 0$ and $(ct - ((n-1)/2^n)\, \varepsilon) > 0$.

Using that $f'_\varepsilon(-1) = f'(-1)$ we claim that

$$\|f'_\varepsilon\|_\infty \geqslant \|f'\|_\infty.$$

The inequality $0 < f_\varepsilon(x) < f(x)$, $-1 \leqslant x < 1$ gives that $\|f_\varepsilon\|_p < \|f\|_p$ and

$$\frac{\|f'_\varepsilon\|_\infty}{\|f_\varepsilon\|_p} > \frac{\|f'\|_\infty}{\|f\|_p}.$$

The proof in the case (c) is completed.

(d) $t \in [(n-2)/(n+2), (n-2)/n]$. Consider $f(x) = c(1+tx)(1-x)^{n-1}$, $c > 0$.

The local extremum of $f'(x)$, $x^*(t) = (2t - n + 2)/nt$ belongs to $[-1, 1)$. In this case $f'(x)$ is decreasing from -1 to $x^*(t)$ and increasing from $x^*(t)$ to 1; $f'(x) \leqslant 0$, $(-1 \leqslant x \leqslant 1)$, so we conclude

$$\|f'\|_\infty = |f'(x^*(t))|.$$

The polynomial

$$p_{x^*}(x) := \frac{(1-x)^{n-1}}{(1-x^*)^n} [(n-1)\, x + 1 - nx^*]$$

satisfies $p_{x^*}(x^*) = 1$, $p'_{x^*}(x^*) = 0$ and $p_{x^*}(x) \geqslant 0$ for $-1 \leqslant x \leqslant 1$.

By making use of $p_{x^*}(x)$ we form a variational polynomial

$$f_\varepsilon(x) = f(x) - \varepsilon p_{x^*}(x).$$

The same arguments as in (c) show that

$$\frac{\|f'_\varepsilon\|_\infty}{\|f_\varepsilon\|_p} > \frac{\|f'\|_\infty}{\|f\|_p}$$

and $f_\varepsilon \in e_{n,1}$ for ε sufficiently small ($\varepsilon > 0$).

Remark 6. Note that the above arguments are inapplicable when $t = -1$. This leads us to consider $c(1-x)^n$ *as a candidate for an extremal*, so let us denote $f_*(x) := (1-x)^n$.

Now we consider an auxiliary extremal problem

$$\sup_{f \in e_{n,1}} \frac{\|f'\|_\infty}{\|f\|_p} = \max\left(\sup_{t \in [(n-2)/n, 1)} \frac{\|[c(1+tx)(1-x)^{n-1}]'\|_\infty}{\|c(1+tx)(1-x)^{n-1}\|_p}, \frac{\|f'_*\|_\infty}{\|f_*\|_p} \right)$$

$$= \max\left(\max_{t \in [(n-2)/n, 1]} \frac{\|[(1+tx)(1-x)^{n-1}]'\|_\infty}{\|(1+tx)(1-x)^{n-1}\|_p}, \frac{\|f'_*\|_\infty}{\|f_*\|_p} \right).$$

Let $f_t(x) := (1+tx)(1-x)^{n-1}$ and $n \geqslant 3$ then

$$\|f'_t\|_\infty = \max(|f'_t(x^*(t))|, |f'_t(-1)|)$$

$$= \max\left(\left(\frac{n-2}{n}\right)^{n-2} \frac{(1+t)^{n-1}}{t^{n-2}}, 2^{n-2} \, |(n-1)\, t - n + 1 + 2t| \right)$$

and from here we conclude

$$\sup_{f \in e_{n,1}} \frac{\|f'\|_\infty}{\|f\|_p}$$

$$= \max\left(\max_{t \in [(n-2)/n, 1]} \frac{|f'_t(x^*(t))|}{\|f_t\|_p}, \max_{t \in [(n-2)/n, 1]} \frac{|f'_t(-1)|}{\|f_t\|_p}, \frac{\|f'_*\|_\infty}{\|f_*\|_p} \right).$$

From the method of the proof of Lemma 6 it easily follows that

$$\max_{t \in [(n-2)/n, 1]} \frac{|f'_t(-1)|}{\|f_t\|_p} \leqslant \max\left[\frac{|f'_1(-1)|}{\|f_1\|_p}, \frac{|f'_{-1}(-1)|}{\|f_{-1}\|_p} \right]$$

$$= \max\left[\frac{|f'_1(-1)|}{\|f_1\|_p}, \frac{\|f'_*\|_\infty}{\|f_*\|_p} \right],$$

$$0 \leqslant p \leqslant \infty.$$

(A) Now we consider for $0 < p < \infty$ the extremal problem

$$\max_{t \in [(n-2)/n, 1]} \frac{|f'_t(x^*(t))|}{\|f_t\|_p} = 2^{1/p} \left(\frac{n-2}{n}\right)^{n-2} \left[\max_{t \in [(n-2)/n, 1]} \frac{(1+t)^{(n-1)p}}{t^{(n-2)p}} \times \frac{1}{\int_{-1}^{1} (1-x)^{(n-1)p} (1+tx)^p \, dx} \right]^{1/p}$$

which is equivalent to the auxiliary extremal problem

$$\min_{t \in [(n-2)/n,\, 1]} \frac{\int_{-1}^{1} (1-x)^{(n-1)p} (1+tx)^p \, dx}{(1+t)^{(n-1)p}} t^{(n-2)p}. \tag{6}$$

In the above considerations we need the restriction $n \geqslant 3$. The case $n=2$ will be studied in the next remark.

Remark 7. The case $n=2$ of the extremal problem (1). Lemma 6 shows that we have to consider only the case when the second zero of our polynomial of degree 2 is less than -1. In this case we have

$$\|f'\|_\infty = |f'(1)|$$

and

$$f_\varepsilon(x) = f(x) - \varepsilon(x-1)^2$$

for sufficiently small ε ($\varepsilon > 0$) shows that f cannot be extremal for $t \in (0, 1)$. So, we obtain that the only candidates for extremals in the case $n=2$ are $c(1-x)^2$, $c(1+x)^2$, $c(1-x)(1+x)$.

Now we continue with the auxiliary problem (6), $n \geqslant 3$.

Denote

$$\Phi_p(t) := \frac{\int_{-1}^{1} (1-x)^{(n-1)p} (1+tx)^p \, dx \, t^{(n-2)p}}{(1+t)^{(n-1)p}}$$

$$D_1(t) := \int_{-1}^{1} (1-x)^{(n-1)p} (1+tx)^{p-1} x \, dx$$

$$D(t) := \int_{-1}^{1} (1-x)^{(n-1)p} (1+tx)^p \, dx$$

$$D_0(t) := \int_{-1}^{1} (1-x)^{(n-1)p} (1+tx)^{p-1} \, dx.$$

Our problem is

$$\min_{t \in [(n-2)/n,\, 1]} \Phi_p(t).$$

For the first derivative of $\Phi_p(t)$ we obtain

$$\frac{1}{p} \frac{\Phi_p'(t)}{\Phi_p(t)} = \frac{\int_{-1}^{1} (1-x)^{(n-1)p} (1+tx)^{p-1} x \, dx}{\int_{-1}^{1} (1-x)^{(n-1)p} (1+tx)^p \, dx} + \frac{n-2-t}{t(1+t)}$$

and

$$\Phi_p'(t)=0 \qquad \text{is equivalent to} \quad \frac{D_1(t)}{D(t)}=\frac{t-n+2}{t(1+t)}.$$

The case $p=1$ is trivial. We have

$$\frac{\Phi_1'(t)}{\Phi_1(t)}=\frac{-t(n^2-n+2)+n^2-n-2}{t(1+t)(n+1-t(n-1))}$$

and

$$\min_{t\in[(n-2)/n,\,1]}\Phi_1(t)=\min\left(\Phi_1\left(\frac{n-2}{n}\right),\,\Phi_1(1)\right)$$

$$\Phi_1(1)=\frac{4}{n(n+1)};$$

$$\Phi_1\left(\frac{n-2}{n}\right)=\frac{(n-2)^{n-2}}{(n-1)^{n-1}}(2n-1)\frac{4}{n(n+1)}.$$

For $n=3$ and $n=4$ we have $\min(\Phi_1((n-2)/n),\,\Phi_1(1))=\Phi_1(1)$.

On the other hand

$$\frac{(n-2)^{n-2}}{(n-1)^{n-1}}(2n-1)=\left(1-\frac{1}{n-1}\right)^{n-2}\left(2+\frac{1}{n-1}\right)\xrightarrow[n\to\infty]{}\frac{2}{e}<1.$$

So for n sufficiently big we have

$$\min\left(\Phi_1\left(\frac{n-2}{n}\right),\,\Phi_1(1)\right)=\Phi_1\left(\frac{n-2}{n}\right).$$

Let us now consider the case $p\in(0,+\infty)$, $p\neq 1$. We have

$$sgn(\Phi_p'(1))=sgn\left[1-\frac{\Gamma[(n-1)\,p+1]\,\Gamma(p)\,2^{(n-1)\,p+p}\Gamma(np+2)}{\left[\begin{array}{r}\Gamma[(n-1)\,p+p+1]\,2^{np+1}\\ \times\Gamma[(n-1)\,p+1]\,\Gamma(p+1)\end{array}\right]}\right]$$

$$=sgn\left[-\frac{1}{2}-\frac{1}{2p}\right]<0.$$

It follows that $\Phi_p'(t) < 0$ in $(1-\delta, 1)$ and $\Phi_p(t)$ is strictly decreasing in $(1-\delta, 1)$.

Now we will study

$$sgn\left\{\Phi_p''(t) : t \in \left(\frac{n-2}{n}, 1\right), \Phi_p'(t) = 0\right\}.$$

We have

$$\frac{1}{p}\frac{\Phi_p'(t)}{\Phi_p(t)} = \frac{D_1(t)}{D(t)} - \frac{t-n+2}{t(1+t)}$$

$$\frac{1}{p}\left[\frac{\Phi_p''(t)}{\Phi_p(t)} - \left(\frac{\Phi_p'(t)}{\Phi_p(t)}\right)^2\right] = \frac{D_1'(t)}{D(t)} - \frac{D_1(t)\, D'(t)}{[D(t)]^2} + \frac{t^2 - 2(n-2)\, t - n + 2}{t^2(1+t)^2}$$

$$D'(t) = p \int_{-1}^{1} (1-x)^{(n-1)p}\, (1+tx)^{p-1}\, x\, dx = pD_1(t)$$

$$\frac{D_0(t) + tD_1(t)}{D(t)} = 1, \qquad \text{and} \qquad \frac{D_0(t)}{D(t)} = \frac{n-1}{1+t}.$$

It follows that

$$sgn\left\{\Phi_p''(t) : t \in \left(\frac{n-2}{n}, 1\right), \Phi_p'(t) = 0\right\}$$

$$= sgn\left\{\frac{D_1'(t)}{D(t)} - \frac{D_1(t)\, D'(t)}{[D(t)]^2} + \frac{t^2 - 2(n-2)\, t - n + 2}{t^2(1+t)^2}\right\}.$$

$$D_1'(t) = (p-1) \int_{-1}^{1} (1-x)^{(n-1)p}\, (1+tx)^{p-2}\, x^2\, dx. \tag{7}$$

The polynomial x^2 can be represented by using Lagrange interpolation at $-1,\ +1,\ -1/t \neq -1, 1$.

$$x^2 = \frac{1}{2(1-t)}\,(1-x)(1+tx) + \frac{1}{2(1+t)}\,(1+x)(1+tx) - \frac{1}{1-t^2}\,(1-x^2)$$

Replacing x^2 with the right-hand side of the above formula in (7) we obtain

$$\begin{aligned} D_1'(t) &= \frac{p-1}{2(1-t)} \int_{-1}^{1} (1-x)^{(n-1)p} (1+tx)^{p-1} (1-x)\, dx \\ &\quad + \frac{p-1}{2(1+t)} \int_{-1}^{1} (1-x)^{(n-1)p} (1+tx)^{p-1} (1+x)\, dx \\ &\quad + \frac{p-1}{1-t^2} \int_{-1}^{1} (1-x)^{(n-1)p} (1+tx)^{p-2} (1-x^2)\, dx \\ &= \frac{p-1}{2(1-t)} [D_0(t) - D_1(t)] + \frac{p-1}{2(1+t)} [D_0(t) + D_1(t)] + R(t) \end{aligned}$$

where after integrating by parts we get

$$R(t) = \frac{1}{t(1-t^2)} [-(n-1)\, p\, D_0(t) - (2+(n-1)\, p)\, D_1(t)].$$

The above formulas give the representation

$$\begin{aligned} \frac{D_1'(t)}{D(t)} &= \frac{(n-1)(p-1)\, t^2 - (n-1)^2\, pt - (p-1)\, t^3}{t^2(1+t)^2\, (1-t)} \\ &\quad + \frac{\left[\begin{array}{c} (n-2)(p-1)\, t^2 - (2+(n-1)\, p)\, t \\ +(n-2)(2+(n-1)\, p) \end{array}\right]}{t^2(1+t)^2\, (1-t)} \end{aligned}$$

$$\frac{D_1(t)}{D(t)} = \frac{t-n+2}{t(1+t)}, \qquad \frac{D_0(t)}{D(t)} = \frac{n-1}{1+t}, \quad \frac{D'(t)}{D(t)} = p\, \frac{D_1(t)}{D(t)}.$$

Finally we obtain

$$\begin{aligned} & sgn \left\{ \Phi_p''(t) : t \in \left(\frac{n-2}{n}, 1 \right), \Phi_p'(t) = 0 \right\} \\ &\quad = sgn \left\{ (p+1)(-tn+n-2) : t \in \left(\frac{n-2}{n}, 1 \right) \right\} < 0. \end{aligned}$$

The above sign inequality shows that $\Phi_p(t)$ may have only one local extremum in $[(n-2)/n, 1]$ and it must be a local maximum.

We make an important for the solution of The problem (1) *conclusion that*

$$\min_{t \in [(n-2)/n,\, 1]} \Phi_p(t) = \min \left[\Phi_p \left(\frac{n-2}{n} \right), \Phi_p(1) \right]. \tag{8}$$

Now, we sum up on the basis of the Eq. (8), that was obtained by studying the extremal problem (6) we claim that

$$\sup_{f \in e_{n,1}} \frac{\|f'\|_\infty}{\|f\|_p} = \max\left(\frac{\left|f'_{(n-2)/n}\left(x^*\left(\frac{n-2}{n}\right)\right)\right|}{\|f_{(n-2)/n}\|_p}, \frac{|f'_1(-1)|}{\|f_1\|_p}, \frac{\|f'_*\|_\infty}{\|f_*\|_p}\right),$$

$$0 < p < \infty. \tag{9}$$

observing that $|f'_1(-1)| > |f'_1(x^*(1))| = |f'_1(4-n)/n)|$.

(B) The case $p = 0$. In this case our extremal problem (6) looks as follows

$$\min_{t \in [0,1]} \Phi_0(t) = \min_{t \in [0,1]} \frac{t^{n-2}}{(1+t)^{n-1}} \exp\left(\frac{1}{2}\int_{-1}^{1} \ln[(1-x)^{n-1}(1+tx)]\,dx\right)$$

and

$$t^2 \frac{\Phi'_0(t)}{\Phi_0(t)} = \frac{(n-1)\,t}{1+t} - \frac{1}{2}\ln(1+t) + \frac{1}{2}\ln(1-t), \qquad t \in [0, 1].$$

Hence, there is $x_0 \in ((n-2)/n, 1)$ such that

$$\Phi'_0(t) \begin{cases} <0, & t \in (x_0, 1) \\ >0, & t \in \left[\dfrac{n-2}{n}, x_0\right). \end{cases}$$

From here

$$\min_{t \in [(n-2)/n, 1]} \Phi_0(t) = \min\left[\Phi_0\left(\frac{n-2}{n}\right), \Phi_0(1)\right]$$

and this completes our consideration in the case $p = 0$.

Taking into account that $f_1(x) = (1+x)(1-x)^{n-1}$ *is a solution of the extremal problem* (3) *and on the basis of* (9) *we claim that*

$$\sup_{f \in \mathcal{P}_n} \frac{\|f'\|_\infty}{\|f\|_p} = \max\left(\frac{|f'_1(-1)|}{\|f_1\|_p}, \frac{\|f'_*\|_\infty}{\|f_*\|_p}\right), \qquad 0 \leqslant p < \infty \tag{10}$$

observing that $f_{(n-2)/n}(x)$ *cannot be extremal* (*see Lemma* 6) *for the extremal problem* (1).

In the next lemma we will compare the values of

$$\frac{|f_1'(-1)|}{\|f_1\|_p} \qquad \text{and} \qquad \frac{\|f_*'\|_\infty}{\|f_*\|_p} = \frac{|f'_{-1}(-1)|}{\|f_{-1}\|_p}$$

when p varies from 0 to ∞.

LEMMA 7. *The following inequalities hold*

$$\text{(a)} \quad \frac{|f_1'(-1)|}{\|f_1\|_p} < \frac{\|f_*'\|_\infty}{\|f_*\|_p}, \qquad 0 \leqslant p < 1;$$

$$\text{(b)} \quad \frac{\|f_*'\|_\infty}{\|f_*\|_p} < \frac{|f_1'(-1)|}{\|f_1\|_p}, \qquad p > 1;$$

$$\text{(c)} \quad \frac{\|f_*'\|_\infty}{\|f_*\|_1} = \frac{|f_1'(-1)|}{\|f_1\|_1}, \qquad p = 1.$$

Proof. If $p = 0$, then

$$\frac{|f_1'(-1)|}{\|f_1\|_0} = \frac{2^{n-1}}{\exp\left(\frac{1}{2}\int_{-1}^{1} n \ln(1-x)\, dx\right)}$$

$$< \frac{n 2^{n-1}}{\exp\left(\frac{1}{2}\int_{-1}^{1} n \ln(1-x)\, dx\right)} = \frac{n}{2} e^n = \frac{\|f_*'\|_\infty}{\|f_*\|_0}.$$

If $0 < p < \infty$, then

$$\frac{|f_1'(-1)|}{\|f_1\|_p} = \frac{1}{2}\left(\frac{\Gamma(pn+2)}{\Gamma(pn-p+1)\,\Gamma(p+1)}\right)^{1/p}$$

and

$$\frac{\|f_*'\|_\infty}{\|f_*\|_p} = \frac{1}{2} n(np+1)^{1/p}.$$

In the case $n = 1$ the statement of the lemma is trivial.

Let $n \geqslant 2$. first we show that for the derivative of

$$\phi_p(v) = \frac{\Gamma(vp+1)}{v^p \Gamma[(v-1)\,p+1]}; \qquad v \in R, \quad v \geqslant 2, \quad 0 < p < \infty$$

we have the formula

$$\frac{1}{p}\frac{\phi_p'(v)}{\phi_p(v)}=\sum_{s=0}^{\infty}\left(\frac{1}{(v-1)\,p+1+s}-\frac{1}{vp+1+s}\right)-\frac{1}{v}$$
$$=\left(\sum_{s=0}^{\infty}\left(\frac{1}{(v-1)\,p+1+s}-\frac{1}{vp+1+s}\right)-\frac{1}{vp}\right)+\frac{1}{vp}-\frac{p}{vp}.$$

Consider the finite sum

$$\sum_{s=0}^{N}\left(\frac{1}{(v-1)\,p+1+s}-\frac{1}{vp+1+s}\right)-\frac{1}{vp}$$
$$=\sum_{s=0}^{N}\left(\frac{1}{vp+1-p+s}-\frac{1}{vp+s}\right)-\frac{1}{vp+N+1}.$$

Taking the limit when $N\to\infty$ we obtain that the both series have the same sum. Hence

$$\frac{1}{p}\frac{\phi_p'(v)}{\phi_p(v)}=\sum_{s=0}^{\infty}\left(\frac{1}{vp+1-p+s}-\frac{1}{vp+s}\right)+\frac{1-p}{vp}$$
$$=\sum_{s=0}^{\infty}-\frac{1-p}{(vp+1-p+s)(vp+s)}+\frac{1-p}{vp}.$$

(a) Let $0<p<1$. We have

$$-\frac{1-p}{(vp+1-p+s)(vp+s)}<-\frac{1-p}{(vp+1+s)(vp+s)}$$
$$=(p-1)\left(\frac{1}{vp+s}-\frac{1}{vp+1+s}\right).$$

The above formula gives that

$$\sum_{s=0}^{\infty}\frac{p-1}{(vp+1-p+s)(vp+s)}<(p-1)\sum_{s=o}^{\infty}\left(\frac{1}{vp+s}-\frac{1}{vp+1+s}\right)=\frac{p-1}{vp}.$$

Hence

$$\frac{1\phi_p'(v)}{p\phi_p(v)}<-\frac{1-p}{vp}+\frac{1-p}{vp}=0$$

and

$$\phi_p(v)<\phi_p(1)=\Gamma(p+1).$$

(b) Let $p > 1$. We have

$$-\frac{1-p}{(vp+1-p+s)(vp+s)} > -\frac{1-p}{(vp+1+s)(vp+s)}$$

$$= (p-1)\left(\frac{1}{vp+s} - \frac{1}{vp+1+s}\right).$$

and from here

$$\sum_{s=0}^{\infty} \frac{p-1}{(vp+1-p+s)(vp+s)} > (p-1) \sum_{s=o}^{\infty} \left(\frac{1}{vp+s} - \frac{1}{vp+1+s}\right) = \frac{p-1}{vp}.$$

Hence

$$\frac{1}{p}\,\frac{\phi_p'(v)}{\phi_p(v)} > -\frac{1-p}{vp} + \frac{1-p}{vp} = 0$$

and

$$\phi_p(v) > \phi_p(1) = \Gamma(p+1).$$

The following representation ends the proof of the lemma

$$\left(\frac{|f_1'(-1)|}{\|f_1\|_p}\right)\left(\frac{\|f_*'\|_\infty}{\|f_*\|_p}\right)^{-1} = \left(\frac{\phi_p(n)}{\Gamma(p+1)}\right)^{1/p}.$$

Note, that

$$\frac{\|f_*'\|_\infty}{\|f_*\|_0} = \frac{n}{2} e^n; \qquad \frac{\|f_*'\|_\infty}{\|f_*\|_p} = \frac{|f_*'(-1)|}{\|f_*\|_p} = \frac{1}{2} n(np+1)^{1/p}.$$

On the basis of (10) *and Lemma* 7 *we obtain the solution of the problem* (1) *that is contained in the next theorem.*

3. THE SOLUTION OF THE PROBLEM (1)

Summing up we have proved the following theorem.

THEOREM 1. *If $f \in \mathcal{P}_n$, then*

$$\sup_{f \in \mathcal{P}_n} \frac{\|f'\|_\infty}{\|f\|_p} = \begin{cases} \dfrac{\|f'_*\|_\infty}{\|f_*\|_0} = \dfrac{n}{2}\, e^n, & p = 0; \\ \dfrac{\|f'_*\|_\infty}{\|f_*\|_p} = \dfrac{1}{2}\, n(np+1)^{1/p}, & 0 < p \leqslant 1; \\ \dfrac{\|f'_*\|_\infty}{\|f_*\|_1} = \dfrac{\|f'_1\|_\infty}{\|f_1\|_1} = \dfrac{n(n+1)}{2}, & p = 1; \\ \dfrac{\|f'_1\|_\infty}{\|f_1\|_p} = \dfrac{1}{2}\left(\dfrac{\Gamma(pn+2)}{\Gamma(pn-p+1)\,\Gamma(p+1)}\right)^{1/p}, & p \geqslant 1. \end{cases}$$

In the case $0 \leqslant p < 1$ the only extremal polynomials are $c(1-x)^n$ and $c(1+x)^n$, $c \in \mathbf{R}$, $c \neq 0$.

In the case $p = 1$ the only extremal polynomials are $c(1-x)^n$, $c(1+x)^n$, $c(1-x)(1+x)^{n-1}$, and $c(1+x)(1-x)^{n-1}$, $c \in \mathbf{R}$, $c \neq 0$.

In the case $p > 1$ the only extremal polynomials are $c(1-x)(1+x)^{n-1}$ and $c(1+x)(1-x)^{n-1}$, $c \in \mathbf{R}$, $c \neq 0$.

By using Stirling's formula, Theorem 1 gives the exact asymptotic of $\|f'\|_\infty / \|f\|_p$, $f \in \mathcal{P}_n$.

COROLLARY 1. *If $f \in \mathcal{P}_n$, then*

$$\sup_{f \in \mathcal{P}_n} \frac{\|f'\|_\infty}{\|f\|_p} = \mathbf{O}(n^{1+1/p}), \qquad n \mapsto \infty, \quad 0 < p < \infty$$

(one may compare with Theorem A).

Remark 8. If $f \in \mathcal{P}_n$ and $f > 0$ on $(-1, 1)$, then $f \in \pi_n$. In other words

$$f(x) = \sum_{k=0}^{n} A_k (1+x)^k (1-x)^{n-k}, \qquad A_k \geqslant 0, \quad k = 0, 1, ..., n. \tag{11}$$

This fact is proved in [12] but it is really contained in an earlier result of Meissner [9]. On the basis of (11) and by using the Fundamental theorem of Linear Programming the exact value $n(n+1)/2$ of $\sup\{\|f'\|_\infty / \|f\|_1, f \in \mathcal{P}_n\}$ is given in [11]. Note, that taking a limit in Theorem 1, when $p \mapsto \infty$ we obtain

$$\begin{aligned}\sup\{\|f'\|_\infty/\|f\|_\infty,\ f\in\mathcal{P}_n\} &= \|f_1'\|_\infty/\|f_1\|_\infty\\ &= \left[(1/2)\left(1-\frac{1}{n}\right)^{-n+1}\right]n\\ &< n\lim_{n\mapsto\infty}\left[(1/2)\left(1-\frac{1}{n}\right)^{-n+1}\right]=\frac{e}{2}n.\end{aligned}$$

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